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On Spacelike Congruences in Riemann-Cartan Space-time

L.N. Katkar · V.K. Patil

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Abstract The theory of spacelike congruences is briefly reviewed. Expressions for the expansion, the rotation and the shear tensor of the spacelike curves and their corresponding natural transport laws in Riemann-Cartan space-time are derived. We consider the Maxwell's equations with torsion and established the Helmholtz theorems on vortex tubes, magnetic flux tubes and electric flux tubes. The fluid in magnetohydrodynamics provides $E^a = 0$, hence counterparts of Helmholtz theorems and the strength of the magnetic flux tube can be measured.

Keywords Spacelike congruences

1 Introduction

The Einstein-Cartan theory of gravitation is simplest generalization of General Relativity (GR) theory. It is based on a space-time with curvature and torsion. In GR theory, the spacelike vectors such as vorticity, magnetic and electric field vectors play an important role in relativistic fluid dynamics and electrodynamics of continuous media. Here we like to focus on the behavior of these spacelike vectors in RC space-time.

In RC space-time, the only kinematical quantity altered by torsion is vorticity. These kinematical quantities have been obtained by Mason and Tsamparlis [1]. Smalley and Krisch [2] have discussed that the spin density affects the free field equation due to torsion contribution to vorticity. We describe the consequences of the vector with torsion in Maxwell's equations.

To discuss the important role of magnetic field in RC space-time, Trautman [3] argue that, a primordial magnetic field could produce global spin alignment near the big bang in anisotropic cosmology models where the effect of spin and torsion may be important.

De Sabbata and Gasperini [4] have modified the Maxwell's equations in which no magnetic current is generated and the total electric charge induced by torsion is zero. In this

L.N. Katkar (✉) · V.K. Patil
Department of Mathematics, Shivaji University, Kolhapur, 416004, India
e-mail: lnk1000@yahoo.com

paper we established the results in magnetohydrodynamics (MHD) approximation where the electric field vanishes.

In Sect. 2, we give general notations, definitions and the results required from the theory of kinematics in RC space-time, in which the kinematics and torsion interact.

In Sect. 3, we describe the Maxwell's equations and its consequences in RC space-time, which are covariantly defined on any differential manifold. The theory of spacelike congruence is briefly introduced and derived the natural transport laws in RC space-time, in Sect. 4. In Sect. 5, we established the relativistic generalization of the Helmholtz theorems. Finally, conclusions are made in Sect. 6.

2 Notation and Kinematics-torsion Interaction

In RC space-time, the most general connection Γ^a_{bc} which is different from the Christoffel symbol $\left\{ {a \atop bc} \right\}$ is asymmetric. Its antisymmetric part defines the torsion tensor as

$$Q_{bc}{}^a = \Gamma^a_{[bc]}. \tag{2.1}$$

It follows from the vanishing of the covariant derivative of the metric tensor that; the connection is expressed as the combination of Christoffel symbol and contortion tensor in the form

$$\Gamma^a_{bc} = \left\{ {a \atop bc} \right\} - K_{bc}{}^a, \tag{2.2}$$

where

$$K_{bc}{}^a = -Q_{bc}{}^a + Q_c{}^a{}_b - Q^a{}_{bc}. \tag{2.3}$$

The Riemann curvature and Ricci tensor are given respectively by

$$R_{abc}{}^d = \Gamma^d_{ac,b} - \Gamma^d_{bc,a} + \Gamma^d_{bt}\Gamma^t_{ac} - \Gamma^d_{at}\Gamma^t_{bc}, \tag{2.4}$$

$$R_{ab} = R_{cab}{}^c. \tag{2.5}$$

The covariant derivative is defined with respect to Christoffel symbol in Riemann geometry, in the same way it is defined by the non-symmetric connection Γ^a_{bc} in RC geometry:

$$\hat{\nabla}_b A^a = A^a{}_{;b} = A^a{}_{,b} + \left\{ {a \atop bc} \right\} A^c, \tag{2.6}$$

$$\nabla_b A^a = A^a{}_{/b} = A^a{}_{,b} + \Gamma^a_{bc} A^c. \tag{2.7}$$

The covariant derivatives in Riemann space-time and RC space-time are related through the relation

$$A^a{}_{/b} = A^a{}_{;b} - K_{bc}{}^a A^c. \tag{2.8}$$

As the consequences of torsion introducing in a space-time, timelike congruences are exploited to study the relativistic analogue of spinning fluid. In Riemann-Cartan kinematics, the connection is the antisymmetric connection which is related to g_{ab} through $g_{ab/c} = 0$. Mason and Tsamparlis [1] have decomposed the covariant derivative of 4-velocity vector u_a as

$$u_{a/b} = \sigma_{ab} + \frac{\theta}{3} h_{ab} + \omega_{ab} + \dot{u}_a u_b + 2h_{ca} Q_{bd}{}^c u^d. \tag{2.9}$$

The signature of the space-time is taken to be $(-,-,-,+)$.

Here

$$h_{ab} = g_{ab} - u_a u_b, \qquad \dot{u}_a = u_{a/b} u^b$$

and

$$\theta_{ab} = \upsilon_{(ab)} = \hat{\theta}_{ab}, \tag{2.10}$$

$$\omega_{ab} = \upsilon_{[ab]} = \hat{\omega}_{ab} + \Omega_{ab}, \tag{2.11}$$

where

$$\upsilon_{ab} = h^c{}_a h^d{}_b u_{c/d} - 2h_{ca} Q_{bd}{}^c u^d, \tag{2.12}$$

$$\Omega_{ab} = h^c{}_a h^d{}_b K_{tcd} u^t. \tag{2.13}$$

The expansion tensor θ_{ab} splits into its trace and trace free part (shear of the fluid)

$$\sigma_{ab} = \theta_{ab} - \frac{\theta}{3} h_{ab} = \hat{\sigma}_{ab}, \tag{2.14}$$

$$\theta = u^a{}_{/a} + Q_a u^a = \hat{\theta}, \tag{2.15}$$

where $\hat{\theta} = u^a{}_{;a}$ and $Q_a = 2Q_{ab}{}^b$.

The kinematical quantities expansion and shear are same in RC and Riemann space-time, only vorticity tensor ω_{ab} alters by the skew tensor Ω_{ab}. The vector part Ω_a related to Ω_{ab} can be defined as

$$\Omega^a = \frac{1}{2} \eta^{abcd} u_b \Omega_{cd}. \tag{2.16}$$

3 The Maxwell's Equations

The minimal coupling of electromagnetic field with torsion in electromagnetic field equations is not gauge invariant. However, Hojman, Rosenbaum, Ryan and Shepley [5] have proposed a hypothesis in which torsion and electromagnetic field are interacting. Mason and Tsamparlis [1] have rewritten the Maxwell's equations with torsion and examine the kinematical role played by torsion. A usual form of Maxwell's equations in general relativity is

$$F^{ab}{}_{;b} = J^a, \qquad F_{[ab;c]} = 0, \tag{3.1}$$

where

$$J^a{}_{;a} = 0. \tag{3.2}$$

The Maxwell's equations can be expressed covariantly or partially on any differential manifold. One can write (3.1) and (3.2) in terms of covariant derivative with respect to the connection $\Gamma^a{}_{bc}$ as

$$\hat{\nabla}_b F^{ab} + Q_{bc}{}^a F^{bc} = J^a, \tag{3.3}$$

$$\hat{\nabla}_a J^a = 0, \tag{3.4}$$

$$F_{[ab/c]} - 2Q_{[ab}{}^t F_{c]t} = 0, \tag{3.5}$$

where $\hat{\nabla}_a = \nabla_a + Q_a$. Operating the permutation tensor η_{arst} on (3.5) and using the identity

$$\eta^{abcd}\eta_{arst} = -3!\delta_r^{[b}\delta_s^c\delta_t^{d]}, \tag{3.6}$$

we readily obtain

$$^*F^{ab}{}_{/b} + F^{bc}T^a_{bc} = 0, \tag{3.7}$$

where

$$T_{bc}{}^a = Q_{bc}{}^a + \delta^a{}_{[b}Q_{c]}, \tag{3.8}$$

is the modified form of the torsion tensor.

The electromagnetic field tensor F^{ab} and the 4-current vector J^a can be expressed in terms of 4-velocity vector u^a as

$$F^{ab} = u^a E^b - u^b E^a - \eta^{abcd}u_c H_d, \tag{3.9}$$

$$^*F^{ab} = u^a H^b - u^b H^a - \eta^{abcd}u_c E_d, \tag{3.10}$$

$$J^a = qu^a + I^a, \tag{3.11}$$

where

$$q = u_a J^a, \qquad I^a = h^a{}_b J^b, \tag{3.12}$$

$$E^a = -F^{ab}u_b, \qquad H^a = \frac{1}{2}\eta^{abcd}u_b F_{cd}, \tag{3.13}$$

where q is the charge density, I^a the conduction current and E^a is the electric field 4-vector.

Now we allow Ellis [6] constraints to obtain divergence equations for E^a and H^a in RC space-time.

(i) Divergence equation for H^a: The Maxwell's equations (3.7) with (3.10) can be decomposed with u^a

$$H^a{}_{/b}h^b{}_a = -(Q_a + T_a)H^a + 2(\omega^a - \Omega^a)E_a, \tag{3.14}$$

where $T^a = 2Q_{abc}u^b u^c$ and is a spacelike vector. In Riemann space-time, Tsamparlis and Mason [7] have studied the barotropic flow of a charged perfect fluid with vanishing magnetic field and shown that if $H^a = 0$ implies that $\hat{\omega}_a E^a = 0$. However, in RC space-time, it is evident that the above result is no longer true, but for $H^a = 0$, we have $(\omega^a - \Omega^a)E_a = 0$.

(ii) Divergence equation for E^a: The Maxwell's equations (3.3) with (3.9) can be decomposed with u^a

$$E^a{}_{/b}h^b_a = -(Q_a + T_a)H^a + q - 2(\omega^a - \Omega^a)H_a. \tag{3.15}$$

An important special case in magnetohydrodynamics (MHD) is the vanishing of electric field as suggested by Ellis [8]. The equation (3.15) with $E^a = 0$ reduces to

$$q = 2(\omega^a - \Omega^a)H_a. \tag{3.16}$$

Similarly, we notice that in RC space-time,

$$\omega^a H_a = 0 \quad \Longleftrightarrow \quad q = -2\omega^a H_a \neq 0, \tag{3.17}$$

while in Riemann space-time, the result reads

$$\hat{\omega}^a H_a = 0 \quad \Longleftrightarrow \quad q = 0. \tag{3.18}$$

This proves the significant role of torsion in RC space-time.

4 Contribution of Torsion to Space like Congruence

The theory of space like congruence in Riemann space-time was first initiated by Greenberg [9]. Further; Tsamparlis and Mason [7] have reviewed the spacelike congruence and discussed the physical interpretation of kinematical parameters in Riemann space-time.

Let us consider the space like curves

$$X^a = X^a(\eta^\alpha, s), \tag{4.1}$$

where the parameter η^α ($\alpha = 1, 2, 3$) represents the particular spacelike curves and s measures the arc length along each curve. The unit tangent vector at any point of this curve is defined as

$$h^a = \frac{dx^a}{ds}, \tag{4.2}$$

where $h^a h_a = -1$ satisfying $h_{a/b} h^a = 0$.

In Riemann space-time, the connecting vector δx^a of two particles on neighboring curves satisfies

$$\underset{h}{\pounds}\, \delta x^a = 0. \tag{4.3}$$

Since the Lie derivative is connection independed, it holds also in RC space-time. By using (4.3) and (2.8) we obtain

$$\underset{h}{\pounds}\, \delta x^a = \delta x^a{}_{/c} h^c - h^a{}_{/c} \delta x^c - 2 Q_{cd}{}^a \delta x^d h^c = 0. \tag{4.4}$$

To observe the deformation of the curves of the congruence, the observer erects a screen orthogonal to the spacelike curve at p, so that the congruence of the curves passes perpendicularly through the screen. Because the connecting vector δx^a need not lie on the screen at p, we introduce the projection operator

$$P_{ab} = g_{ab} - w_a w_b + h_a h_b, \tag{4.5}$$

with $P_{ab} w^a = 0$, $P_{ab} h^a = 0$ and the spacelike vector h^a is orthogonal to 4-velocity vector w^a at p, thus $w^a w_a = 1$, $w_a h^a = 0$.

The orthogonal connecting vector at p is given by

$$\delta_\perp x^a = P^a{}_b \delta x^b. \tag{4.6}$$

This vector is orthogonal to vectors w^a and h^a. Hence any connecting vector δx^a can be decomposed in the manner

$$\delta x^a = \delta_\perp x^a + (\delta x^c w_c) w^a - (\delta x^c h_c) h^a. \tag{4.7}$$

The quantity $\delta_\perp v^a$ which expresses the rate of change of two spacelike curves, then

$$\delta_\perp v^a = P^a{}_b(\delta_\perp x^b)^*, \tag{4.8}$$

where $\overset{*}{A}{}^a = \frac{DA^a}{ds} = A^a{}_{/b}h^b$.

With the aid of (4.4) and (4.5), the expression (4.8) gives

$$\delta_\perp v^a = P^a{}_c(h^c{}_{/d} + 2Q_{td}{}^c h^t)\delta x^d - P^a{}_b \overset{*}{w}{}^b w_c \delta x^c + P^a{}_b \overset{*}{h}{}^b h_c \delta x^c. \tag{4.9}$$

Now by using the definition $\overset{o}{h}_a = h^a{}_{/b}w^b$ and (4.7), the desire form of (4.9) is

$$\delta_\perp v^a = P^a{}_b P^c{}_d(h^b{}_{/c} - 2Q_{ct}{}^b h^t)\delta x^d + P^a{}_b(\overset{o}{h}{}^b - \overset{*}{w}{}^b - 2Q_{ct}{}^b h^t w^c)w_d \delta x^d. \tag{4.10}$$

In (4.10), the presence of the second term $B^{ad}\delta x_d$, where

$$B^{ad} = P^a{}_b(\overset{o}{h}{}^b - \overset{*}{w}{}^b - 2Q_{ct}{}^b h^t w^c)w^d, \tag{4.11}$$

is crucial. Except at the given point p, the motion of the second observer, displayed a distance ds along the congruence has to be specified. In other words, a natural transport laws for w^a must be specified and which appears in the form

$$\overset{*}{w}{}^a = \overset{o}{h}{}^a - \overset{o}{h}{}^b w_b w^a + \overset{*}{h}_b w^b h^a - 2Q_{ct}{}^a h^t w^c + 2Q_{ct}{}^b h^t w^c w_b w^a - 2Q_{ct}{}^b h^t w^c h_b h^a. \tag{4.12}$$

With this transport law the term B^{ad} vanishes, this ensure that the rotation and shear are to lie in the two-space of the screen.

With $B_{ab} = 0$, (4.10) reduces to

$$\delta_\perp v^a = P^a{}_b P^c{}_d(h^b{}_{/c} - 2Q_{ct}{}^b h^t)\delta x^d. \tag{4.13}$$

By defining the operator

$$A_{ab} = P^c{}_a P^d{}_b h_{c/d} - 2P^c{}_a P^d{}_b Q_{dtc}h^t, \tag{4.14}$$

we have

$$\delta_\perp v_a = A_{ab}\delta x^b = A_{ab}\delta_\perp x^b. \tag{4.15}$$

Further the spacelike congruences in RC space-time and Riemann space-time are related through the relation (2.8)

$$A_{ab} = \tilde{A}_{ab} + \Omega_{ab}, \tag{4.16}$$

where

$$\tilde{A}_{ab} = P^c{}_a P^d{}_b h_{c;d}, \tag{4.17}$$

$$\Omega_{ab} = P^c{}_a P^d{}_b K_{tcd}h^t. \tag{4.18}$$

We decompose A_{ab} into its irreducible parts as follows:

$$A_{ab} = \Re_{ab} + \frac{1}{2}\Theta P_{ab} + \Im_{ab}, \tag{4.19}$$

where

$$\Re_{ab} = A_{[ab]} = \tilde{\Re}_{ab} + \Omega_{ab}. \tag{4.20}$$

Because of $K_{a(bc)} = 0$, $\Omega_{(ab)} = 0$ we define

$$\Theta = A^a{}_{.a} = h^a{}_{/a} - h_{a/b} w^a w^b + (Q_a + T_a) h^a = \tilde{\Theta}, \tag{4.21}$$

$$\Im_{ab} = A_{(ab)} - \frac{1}{2} A^c{}_c P_{ab} = \tilde{\Im}_{ab}. \tag{4.22}$$

Clearly

$$\Re_{ab} w^a = 0, \qquad \Re_{\mathrm{ab}} h^a = 0, \tag{4.23}$$

$$\Im_{ab} w^a = 0, \qquad \Im_{\mathrm{ab}} h^a = 0. \tag{4.24}$$

On evaluation the expression for rotation tensor $\Re_{ab}$, the expression Θ and the shear tensor $\Im_{ab}$, only rotation tensor alters by torsion through the skew tensor Ω_{ab} in RC space-time.

Finally, δA be the crossectional area of the flux tube subtended by the spacelike curves as they pass through the screen. The terms $\Re_{ab} \neq 0$ and $\Im_{ab} \neq 0$ leave the crossectional area δA invariant, and then we define

$$\frac{(\delta A)^{\bullet}}{\delta A} = \Theta. \tag{4.25}$$

5 Physical Interpretation

The physical interpretation of the spacelike congruences as we developed in RC space-time is aided if we introduced the concept of vortex tubes, magnetic flux tubes and electric flux tubes.

5.1 Vortex Tubes

We first consider the Helmholtz theorems on vortex tubes in Newtonian theory. As the torsion is connected with the covariant derivative of a vector, it vanishes in Newtonian theory.

In Newtonian theory, it follows from (4.25) and (4.21) with $h^a = \frac{\omega^a}{\omega}$, that

$$\frac{D}{ds} \ln(\omega \delta A) = \frac{1}{\omega} \omega^a{}_{,a}, \tag{5.1}$$

where δA is the cross-sectional area of the vortex tube. But $\omega^a{}_{,a} = 0$ as shown by Greenberg [9], then $\omega \delta A$ is constant along the vortex tubes and it can be measured the strength of vortex tube and is called the first Helmholtz theorem.

For the extension of this theorem in RC space-time, consider an observer at any point p on the vortex lines of a vortex tube. With the aid of (4.25) with $h^a = \frac{\omega^a}{\omega}$ and (4.21), we have

$$\frac{D}{ds} \ln(\omega \delta A) = \frac{1}{\omega} \big[\omega^a{}_{,b} h^b_a + (Q_a + T_a) \omega^a \big]. \tag{5.2}$$

The relativistic analogues of the second and third Helmholtz theorem depend on the fluid or flow chosen.

5.2 Magnetic Flux Tubes

In RC space-time, the Helmholtz theorem for vortex tubes can be derived for magnetic flux tubes. For a commoving observer at any point p on the magnetic field lines of a magnetic flux tubes, it follows from (4.25) and (4.21) with $h^a = \frac{H^a}{H}$ that

$$\frac{D}{ds}\ln(H\delta A) = \frac{1}{H}[H^a{}_{;b}h^b{}_a + (Q_a + T_a)H^a], \tag{5.3}$$

where $H\delta A$ is the cross-sectional area of the flux tube along the magnetic field lines. Hence by using divergence equation (3.14), we have

$$\frac{D}{ds}\ln(H\delta A) = -\frac{2}{H}(\omega^a - \Omega^a)E_a. \tag{5.4}$$

In Riemann space-time, if the assumption in which the vorticity $\hat{\omega}_a$ and the electric field vector E^a are orthogonal, then $H\delta A$ is constant and can be taken as a measure of the strength of the tube. In fact in RC space-time, if $\omega^a E_a = 0$, then

$$\frac{D}{ds}\ln(H\delta A) = -\frac{2}{H}(\Omega^a E_a) \neq 0. \tag{5.5}$$

Hence above result is no longer true for non-vanishing torsion.

To obtain the counterparts of second and third Helmholtz theorem, the fluid in MHD provides $E^a = 0$ and the magnetic field lines are material lines. Then (5.5) reduces to

$$\frac{D}{ds}\ln(H\delta A) = 0, \tag{5.6}$$

showing that the strength of the tube can be measured in RC space-time.

5.3 Electric Flux Tubes

In this part of the section, we established the Helmholtz theorem for electric flux tube in RC space-time.

For a commoving observer at any point p on one of the electric field lines of an electric flux tube, from (4.25) and (4.21) with $h^a = \frac{E^a}{E}$ and Maxwell's equation (3.15), we have

$$\frac{D}{ds}\ln(E\delta A) = \frac{1}{E}[q - (\omega^a - \Omega^a)H_a]. \tag{5.7}$$

We infer under more general condition that the fluid provided with $q - (\omega^a - \Omega^a)H_a = 0$, and then $E\delta A$ is constant along the flux tube, and can be taken as a measure of the strength of the tube.

6 Conclusion

If Maxwell's electromagnetic field equations and the spacelike congruences are valid in a Riemann-Cartan space-time, then a study of Helmholtz theorems on vortex tubes, magnetic field tubes and electric field tubes presents no difficulties, because the Maxwell's equations and spacelike congruences are connection independent.

In order to modify the Helmholtz theorem in RC space-time according to (5.2), (5.4) and (5.7), this modification, however, still preserve the essential characters as the fluid or flow provided. Moreover torsion is not coupled directly with electric field; (5.6) shows that the strength of magnetic flux tube $H\delta A$ remains constant during the motion in magnetohydrodynamics. Our modified form of Helmholtz theorems reduces to usual form in Riemann space-time when torsion vanishes.

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